HW ONE, MTH 512, Spring 2015, Due date: March 7, 2015 at 3pm

Ayman Badawi

QUESTION 1. (i) Let $v_1, v_2, ..., v_n$ be independent elements of a vector space V. Assume there is a $w \in V$ such that $w \notin span\{v_1, v_2, ..., v_n\}$. Prove that $w, v_1, v_2, ..., v_n$ are linearly independent.

(ii) Given v_1, v_2, v_3 are independent elements of a vector space V. Given a, b, c, d are some nonzero constants. Prove that $av_1, bv_1 + cv_2, v_1 + v_2 + dv_3$ are linearly independent elements of V.

(iii) Let $M = \{f(x) \in P_7 \mid f(1) = 0$ or $f(-1) = 0\}$. Is M a subspace of P_7? prove or disprove.[Recall that P_n is the set of all polynomials of degree $< n$].

(iv) Let $M = \{f(x) \in P_7 \mid f(1) = 0$ and $f(-1) = 0\}$. Is M a subspace of P_7? prove or disprove.

(v) Let D be a subspace of a vector space V such that $D \neq V$. Given $dim(V) = m$. Prove that $dim(D) < m$.

(vi) Are $(2, 0, -2, 3), (-2, 0, 3, -3), (0, 0, 2, 0)$ independent elements of R^3? Show the work.

(vii) Let $v_1, v_2, ..., v_m$ be any elements in a vector space V. Prove that span$\{v_1, ..., v_m\}$ is a subspace of V.

(viii) Given $v_1 = (0, 1, 2), v_2 = (1, 0, 4)$ are independent elements of R^3. Is $w = (3, 2, 16) \in span\{v_1, v_2\}$? If no, show the work. If yes, then find a, b such that $w = av_1 + bv_2$.

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T OR F

Ayman Badawi

QUESTION 1. just write T for true or F for false

(i) If A is a 4×4 matrix and $C_A(x) = x^2(x-3)^2$, then the system $AX = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions.

(ii) If A is a 4×4 matrix and 1 is an eigenvalue of A, then there is a nonzero 4×10 matrix B, such that $AB = B$.

(iii) It is impossible to construct a linear transformation $T : R^2 \to R^4$ such that $dim(Range(T)) = 3$.

(iv) If A is a 4×5 matrix and $Rank(A) = 4$, then the system $AX = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ has infinitely many solutions

(v) If 6 nonzero distinct points in R^4 are given, then 4 points of the given points are independent.

(vi) If $T : R^3 \to R$ is a linear transformation and $T(1, 4, 7) = \pi$, then $dim(Ker(T)) = 2$.

(vii) $span\{x^2+1, 6x+3, x^2+2x+2\} = P_3$ (note that my definition of P_3 is the set of all polynomials of degree strictly less than 3, and hence $dim(P_3) = 3$)

(viii) If A is a 3×3 matrix such that $det(A) = 0$, then the system $AX = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$ has infinitely many solutions

(ix) If A is a 3×3 invertible matrix, then A is diagnolizable.

(x) If A is a 4×4 matrix and the system $AX = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 7 \end{bmatrix}$ is inconsistent (i.e., it has no solution), then the system $AX = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions

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HW three, MTH 512, Spring 2015, Due date: March 28, 2015 at 3pm

Ayman Badawi

QUESTION 1. (i) Let $T : V \longrightarrow W$ be a linear transformation (of course V and W are vector spaces). Let $d \in Range(T)$. Hence $T(v_1) = d$ for some $v_1 \in V$. Let $M = \{v \in V \mid T(v) = d\}$. Prove that $M = v_1 + Ker(T)$. i.e., show that $M = \{v_1 + a \mid a \in Ker(T)\}$.

(ii) Let $T : P_3 \longrightarrow R$ be a linear transformation such that $T(5x) = 0$, $T(-3x^2) = 9$, $T(10) = 20$.

a. Find a basis for Kert(T).

b. Find $T(6x^2 + 9x - 13)$.

c. Describe all elements in V such that each has image (under T) equals to 13.

(iii) Let $T : V \longrightarrow W$ be a linear transformation, and F be a subspace of V. Prove that $T(F)$ is a subspace of W.

(iv) Let $M = \{f(x) \in P_3 \mid \int_0^1 f(x)\, dx = 0\}$. Prove that M is a subspace of P_3. Find a basis for M.

(v) Assume K, L are proper subspaces V such that $K \not\subseteq L$ and $L \not\subseteq K$. Prove that $K \cap L$ is a subspace of V, but $K \cup L$ is never a subspace of V.

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HW four, MTH 512, Spring 2015, Due date: March 28, 2015 at 3pm

Ayman Badawi

QUESTION 1. Let $T : R^{2\times 2} \longrightarrow P_3$ such that $T(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}) = (a_1 + a_4)x^2 + (a_3 + a_2)$.

(i) Show that T is a linear transformation.

(ii) Find the standard matrix representation of T.

(iii) Find a basis for Range(T).

(iv) Find a basis for Ker(T).

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HW Five, MTH 512, Spring 2015, Due date: April 18, 2015 at 3:20pm

Ayman Badawi

QUESTION 1. Let $V = C(-\infty, \infty)$ be the set of all functions such that for every $n \geq 1$, the n^{th} derivative is continuous on R. Let $T : V \longrightarrow V$ such that $T(f(x)) = -f^{(2)}(x) - 2f'(x)$ (where $f^{(2)}(x)$ indicates the second derivative of $f(x)$). It is trivial to show that T is a linear transformation. So do not show that.

(i) Prove that T has infinitely many eigenvalues.[Hint: in basic Diff. Eq course, to solve $y^{(2)} + ay' + by = 0$, we set the equation $m^2 + am + b = 0$, if m_1, m_2 are two real distinct solutions, then the solution to the Diff. Eq. is $span\{e^{m_1 x}, e^{m_2 x}\}$, what about if $m_1 = m_2$?, what about if m_1 is imaginary number? so it is about time, being a graduate student, to review your basic Diff. Eq. course]

(ii) For each eigenvalue a of T, write E_a as a span of some basis.

QUESTION 2. Let $T : R^3 \longrightarrow R^3$ such that $T(a_1, a_2, a_3) = (-3a_3, a_1 + 5a_3, a_2 - a_3)$. It is trivial to show that T is a linear transformation (do not show that).

(i) Find all eigenvalues of T (I believe that 1 is one of the eigenvalues).

(ii) For each eigenvalue a, E_a as a span of some basis.

(iii) Is T diagnolizable? If yes, tell me why. If no, tell me why

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HW six, MTH 512, Spring 2015, Due date: April 29, 2015 at 6:00 pm

Ayman Badawi

QUESTION 1. Let V be a vector space and W_1, W_2 are subspaces of V. Then $W_1 + W_2 = \{a + b \mid a \in W_1, b \in W_2\}$.

(i) Prove that $W_1 + W_2$ is a subspace of V.

(ii) Assume $dim(W_1) = n, dim(W_2) = m$. Prove that $dim(W_1 + W_2) = dim(w_1) + dim(W_2) - dim(W_1 \cap W_2)$.

(iii) Assume $dim(V) = 5, dim(W_1) = 3, dim(W_2) = 4$ and $W_1 \not\subseteq W_2$. Find $dim(W_1 + W_2)$ and $dim(W_1 \cap W_2)$.

(iv) Let $F = \{(a, a + b, 4b, 0) \mid a, b \in R\}$ and $K = \{(c, 2c + d, 4c - d, 2d), d \in R\}$. Clearly F and K are subspaces of R^4. Write $W_1 + W_2$ as a span of some basis.

(v) Let $T : V \longrightarrow V$ be a linear transformation. Let a be a nonzero number. It is trivial to check that aT is a linear transformation. If b is an eigenvalue of T. Prove that ab is an eigenvalue of aT (Hence if b is an eigenvalue of a matrix A, then ab is an eigenvalue of aA)

(vi) Let $T : V \longrightarrow V$ be a linear transformation such that $dim(V) = n < \infty$. Prove that T is bijection if and only if T is one-to-one (i.e., injective)

(vii) Let $T : R^7 \longrightarrow R^7$ be a linear transformation such that 9 is an eigenvalue of T and $dim(E_9) = 6$. Prove that either $T - 4I$ or $T - 5I$ is a bijection linear transformation from V ONTO V. [Hint: A deceiving hint is to observe that 4 + 5 = 9 :)))]

QUESTION 2. The Fibonacci sequence $F_0, F_1, F_2, ...$ is defined by $F_0 = 0, F_1 = 1; F_2 = 1$; and $F_n = F_{n-2} + F_{n-1}$. Use the concept of linear transformation to prove that $F_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right)$. In order to do that, follow the following steps.

(i) Let $T : R^2 \longrightarrow R^2$ such that $T(a, b) = (b, a + b)$. Then T is clearly a linear transformation (do not show that)

(ii) Show that $T^n(0, 1) = (F_n, F_{n+1})$[use math induction. First show it is true for $n = 2$. Assume it is true for $n = m$. Prove it for $n = m + 1$]

(iii) Show that T is diagnolizable.

(iv) Now use (iii), to find $T^n(0, 1)$. [note that if $M = QDQ^{-1}$, where D is a diagonal matrix, then $T^n(0, 1) = M^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = QD^nQ^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, the Formula for F_n should be observed.

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HW 7, MTH 512, Spring 2015, Due date: May 9, 2015 at 3:00 pm

Ayman Badawi

QUESTION 1. (i) The companion matrix of a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ is the $n \times n$ matrix F such that its rows determined as follows $R_1 = (0, 0, ..., -a_0)$, $R_2 = (1, 0, 0, ..., -a_1)$, $R_3 = (0, 1, 0, 0, ..., -a_2)$, $R_n = (0, 0, 0, ..., 1, -a_{n-1})$. Prove that $m_F(x) = C_F(x) = f(x)$. [Hint use the fact that every polynomial over the complex is linearly factored, to find $C_F(x)$ use the last column of $xI_n - F$ and use $C_F(x) = det(xI_n - F)$]

(ii) Let F be the companion matrix of $f(x) = (x-3)^3(x+2)^2$. Then F is similar to a matrix J in Jordan-form. Find J.

(iii) Recall our definition of N_m. Let $A = N_5^2$. Find the Jordan form of A. Let $B = N_4^3$. Find the Jordan form of B.

(iv) Let $A = \begin{bmatrix} 0 & -16 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$. Find the Jordan-form of A.

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HW 8, MTH 512, Spring 2015, Due date: May 16, 2015 at 3:00 pm

Ayman Badawi

QUESTION 1. (i) Let $<,>$ be an inner product on a vector space V. Suppose that v_1, v_2 are NONZERO elements of V such that $v_1 \perp v_2$. Prove that v_1, v_2 are independent.

(ii) Let $<,>$ be the normal dot product on R^3. Find two nonzero independent elements in R^3 that are not orthogonal.

(iii) Let E be a subspace of a vector space V and assume that $<,>$ is an inner product on V. Then $E^{\perp} = \{x \in V \mid x \perp e$ for every $e \in E\}$.

a. Prove that $E^{\perp}$ is a subspace of V.

b. Find $E \cap E^{\perp}$.

c. Let $B = \{w_1, ..., w_k\}$ be a basis for E. We know $E = span\{w_1, ..., w_k\}$. Prove that $E^{\perp} = \{y \in V \mid y \perp w_i$ for each i, $1 \leq i \leq k\}$.

d. Assume that $B = \{w_1, ..., w_k\}$ is an orthogonal a basis for E. Let $v \in E^{\perp}$ such that $v \neq O_V$. Prove that $v = w + c_1w_1 + c_2w_2 + ... + c_kw_k$ for some nonzero $w \notin E$ and for some constants$c_1, ..., c_k \in R$.

e. (converse of (d)). Let $B = \{w_1, ..., w_k\}$ be an orthogonal basis for E. Let $w \notin E$. Prove that $v = w - proj_{w_1}^{(w)} - proj_{w_2}^{(w)} - \cdots - proj_{w_k}^{(w)} \in E^{\perp}$ (Note that in this part, it seems that I calculated the $c_1, ..., c_k$ in part (d), also this part, give you an algorithm on how to construct an orthogonal basis for $E^{\perp}$).

f. If V is finite dimensional, say has dimension n, prove that $dim(E^{\perp}) = n - dim(E)$. Hence $E + E^{\perp} = V$ and $dim(E + E^{\perp}) = n$[Hint: use part (e)]

(iv) Let $<,>$ be the inner product on $V = C[0, \pi/2]$ defined by $< f_1(x), f_2(x) >= \int_0^{\pi/2} f_1(x)f_2(x)\, dx$. Let $E = span\{2, sin(x), cos(x)\}$. Find an orthogonal basis for E.(see class notes)

(v) Let A be a nonzero $n \times m$ matrix. Let $E = Row(A)$. We know that Row(A) is a subspace of R^m. Under the normal dot product on R^m, it is trivial to see (by staring and by part (c)) that $Nul(A) = E^{\perp}$. Use this fact, to find all vectors in R^4 that are orthogonal to the vectors $v_1 = (1, -1, 2, 4), v_2 = (-1, 1, -1, 8) \in R^4$ (here we still using the normal dot product on R^4).[Hint: form a matrix A with two rows v_1, v_2. Then $Nul(A)$ is the set of all such vectors

(vi) Let Let $<,>$ be an inner product on a vector space V. Let $u, v \in V$. Prove that $(||u + v||)^2 + (||u - v||)^2 = 2||u||^2 + 2||v||^2$.

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